D-Stability and Real and Complex Quadratic Forms

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ABSTRACT

The problem of characterizing the class of D-stable matrices has remained unsolved since its suggestion in a paper by Arrow and McManus [1] in 1958. In this note we present a necessary condition involving real quadratic forms which is not sufficient and a sufficient condition involving complex quadratic forms which is not necessary.

Denote by $M_n(F)$ the class of n by n matrices over the field F, where F may be either the real numbers R or the complx numbers C. Let D_n be the class of diagonal matrices with real positive diagonal entries. We denote the spectrum of $A \in M_n(F)$ by $\sigma(A)$. Then $A \in M_n(F)$ is said to be D-stable if $D \in D_n$ implies $\mathrm{Re}(\lambda) > 0$ for all $\lambda \in \sigma(DA)$. We shall denote the class of D-stable elements of $M_n(F)$ by $DL_n(F)$ and their Euclidean closure by $\overline{DL_n(F)}$.

We shall not review the known facts about D-stability [1,6,7,8,9] except to mention that the broadest known sufficient condition is as follows. If there is a $D \in D_n$ such that $DA + A^*D$ is positive definite, then $A \in DL_n(C)$. Several of the other sufficient conditions are known to be special cases of this one [4,5,7]. However, this condition is a special case of the sufficient condition given by Theorem 2 below.

We proceed to relate D-stability to quadratic forms. Theorem 1 points out a necessary condition involving real quadratic forms which Example 1 shows is not sufficient. Theorem 2 gives a new sufficient condition involving complex quadratic forms which Example 2 shows is not necessary.

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THEOREM 1. *Suppose $A \in DL_n(R)$. Then for each $0 \neq x \in R^n$, there is a $D \in D_n$ (which depends on x) such that*

$$x^T DAx \geqslant 0.$$

Proof. Suppose to the contrary that there is a nonempty (necessarily open) set $S \subset R^n$ such that for all $D \in D_n$, $x \in S$

$$x^T DAx < 0. \tag{*}$$

Suppose notationally that

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad Ax = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}.$$

Since S is open, we may choose $x \in S$ such that $x_i y_i \neq 0$, $i = 1, \ldots, n$. Our assumption (*) now becomes

$$\sum_{i=1}^{n} y_i d_i x_i < 0 \tag{**}$$

for all choices $d_1, \ldots, d_n > 0$. Thus it must be that each $y_i x_i < 0$, $i = 1, \ldots, n$. It then follows that for any negative number λ, the system

$$d_i y_i = \lambda x_i, \quad i = 1, \ldots, n,$$

may be solved by choice of $d_1, \ldots, d_n > 0$. Such a solution is equivalent to the statement

$$DAx = \lambda x,$$

which means $\lambda \in \sigma(DA)$. The existence of a negative member of $\sigma(DA)$ contradicts $A \in DL_n(R)$ and completes the proof.

Note. In the statement of Theorem 1, "$\geqslant$" may not be replaced by "$>$" since $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \in DL_2(R)$.

Let $\overline{D_n}$ denote the Euclidean closure of D_n. By a straightforward compactness argument, it follows from Theorem 1 that

COROLLARY 1.1. *If $A \in \overline{DL_n(R)}$, then for each $0 \neq x \in R^n$ there is a $D \in \overline{D_n}$ (which depends on x) such that $x^TDx > 0$ and $x^TDAx \geqslant 0$.*

In a different context Fiedler and Ptak have studied the class defined by the necessary condition of Corollary 1.1 and related it to the class P_0 which is defined as follows [2].

DEFINITION. A is an element of the class P_0 if and only if all the principal minors of $A \in M_n(R)$ are nonnegative.

LEMMA. [2] *For $A \in M_n(R)$, the following are equivalent:*

(i) $A \in P_0$;

(ii) *for each $0 \neq x \in R^n$, there is a $D \in D_n$ (depending on x) such that $x^TDx > 0$ and $x^TDAx \geqslant 0$;*

(iii) *each real eigenvalue of each principal submatrix of A is nonnegative.*

It then follows from Corollary 1.1 and the lemma that

COROLLARY 1.2. $\overline{DL_n(R)} \subseteq P_0$.

This containment is implicit in [7].

EXAMPLE 1. The converse of Theorem 1 is false (as well as those of the corollaries). Let

$$A = \begin{bmatrix} 1 & 0 & 27 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then $4 \in \sigma(A)$, and since $\mathrm{Tr}(A) < 4$ $A \notin \overline{DL_3(R)}$. However, since all principal minors of A are positive, it follows from Theorem 1.1 of [2] that for each $0 \neq x \in R^n$, there is a $D \in D_n$ such that

$$x^T DAx > 0.$$

Although the containment in Corollary 1.2 is proper, it suggests an alternative to studying $\overline{DL_n(R)}$ would be found in the following class.

DEFINITION. Let $P_0^- =$

$$\{A \in P_0\text{: there is a } \lambda \in \sigma(A) \text{ with } \mathrm{Re}(\lambda) < 0\}.$$

The lemma points out that if $A \in P_0$ and $\mathrm{Re}(\lambda) < 0$ for some $\lambda \in \sigma(A)$, then $\mathrm{Im}(\lambda) \neq 0$.

REMARK. $\overline{DL_n(R)} = P_0 - D_n P_0^-$.

Thus to characterize $\overline{DL_n(R)}$ it would suffice to obtain a satisfactory characterization of P_0^-.

THEOREM 2. *If $A \in M_n(C)$ is such that for each $0 \neq x \in C^n$ there is a $D \in D_n$ (depending on x) such that*

$$\mathrm{Re}(x^* DAx) > 0,$$

then $A \in DL_n(C)$.

Proof. Suppose $A \notin DL_n(C)$. Then there is an $E \in D_n$ such that there is a $\lambda \in \sigma(EA)$ with $\mathrm{Re}(\lambda) \leqslant 0$. Suppose

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is such that

$$EAx = \lambda x,$$

and let

$$Ax = y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

It then follows that $\mathrm{Re}(y_i\bar{x}_i)\leqslant 0, i=1,\dots,n$. Now

$$x^*DAx=\sum_{i=1}^{n} d_i y_i \bar{x}_i,$$

where $D=\mathrm{diag}\{d_1,\dots,d_n\}$ and the right-hand side clearly has nonpositive real part for all $D\in D_n$. The statement of the theorem is the contrapositive of what has been shown.

By continuity we have also that

COROLLARY 2.1. *If $A\in M_n(C)$ is such that for each $x\in C^n$ there is a $D\in D_n$ (depending on x) such that*

$$\mathrm{Re}(x^*DAx)\geqslant 0,$$

then $A\in\overline{DL_n(C)}$.

EAMPLE 2. The converses of Theorem 2 and Corollary 2.1 are false. Let

$$A=\begin{bmatrix} 1 & 0 & -150 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad\text{and}\quad x=\begin{bmatrix} -9+20i \\ -4-\ 4i \\ 4+\ 2i \end{bmatrix}.$$

Then it may be verified [3,6] that $A\in DL_3(C)$. However, if

$$Ax=y=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

then

$$\mathrm{Re}(y_i\bar{x}_i)\leqslant 0,\quad i=1,2,3, \text{ and } \mathrm{Re}(y_i\bar{x}_i)<0,\quad i=1,2.$$

Thus

$$\mathrm{Re}(x^*DAx)<0$$

for all $D\in D_n$.

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